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SOME COMPLEX EQUATIONS ARISING IN HELE SHAW FLOW

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Abstract—In this note, we treat a two-dimensional moving boudary value problem by a complex differential equation. We obtain an estimate on the C^α norm of the moving boundary.

1. INTRODUCTION

This paper is devoted to the following complex differential equation:

$$\begin{aligned} \operatorname{Re}\left[\frac{\partial f(\zeta,t)}{\partial t}\bar{\zeta}\frac{\partial \bar{f}(\zeta,t)}{\partial \bar{\zeta}}\right] = 1 \qquad &\text{for } |\zeta| < 1, \\ f(\zeta,0) = f(\zeta) \qquad &\text{for } \zeta \in \Delta, \text{ the unit disk.} \end{aligned} \tag{1.1}$$

This equation was used in [1] by Richardson to describe the solution of a two-dimensional moving boundary value problem. Under the restrictive assumption that $f(.,0)$ is analytic and univalent in a neighbourhood of the unit disk, Vinogadov and Kufarev [2] proved the existence of a solution of (1.1). In 1984, a simpler proof of existence in some special cases was given by Gustafason [3].

The main result of this paper is the following.

THEOREM 1.1. *Suppose*

(a) $f(.)$ *is analytic in a neighborhood of the unit disk,*
(b) $f(.,0) \in C^{1,\alpha}(\bar{\Delta})$,
(c) $|f_\zeta(.)| > \delta > 0$ *for* $\zeta \in \bar{\Delta}$.
(1) *Then problem (1.1) has exactly one solution in* $\Delta \times [0,T]$, *where* T *is a sufficiently small positive number.*
(2) *There are constants* C_1 *independent of* f *and* C_2 *depending on* $f(.,0)$ *such that*

$$\|f(.,t)\|^2_{C^{1/2}_\Delta} \le C_1 t + C_2.$$

The second statement of the theorem gives a quantitative description of the moving boundary, which is lacking in the references.

2. PROOF OF THEOREM 1.1

In the following proof of Theorem 1.1, we use $u(\zeta,t)$ to denote $\frac{\partial f}{\partial \zeta}(\zeta,t)$ and $p(u)(\zeta,t)$ to denote $\frac{1}{2\pi i}\int_{\partial\Delta}\frac{1}{|u(z,t)|^2}\frac{z+\zeta}{z-\zeta}\frac{dz}{z}$, which is the Poisson integral of $1/|u|^2$. In terms of u and $p(u)$, (1.1) can be easily reformulated as:

$$\begin{aligned} u_t - (\zeta u p(u))_\zeta = 0, \qquad &\zeta \in \Delta \\ u|_{t=0} = u_0, & \end{aligned} \tag{2.1}$$

where $u_0 = f_\zeta(\zeta)$. The proof of Theorem 1.1 relies heavily on the following.

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PROPOSITION 2.1. *Suppose u is a solution of (2.1) such that $u(t,.)$ is analytic in Δ. Then*

$$\int_{\partial\Delta} |u(\zeta,t)|^2\, dA \le \int_{\partial\Delta} |u_0(\zeta)|^2\, dA + 4\pi t.$$

PROOF. Let Δ be the unit disk and Δ_r be the disk centered at 0 with radius r. Then from equation 2.1 we know

$$\int_{\Delta-\Delta_r} u_t\bar{u}\, dA - \int_{\Delta-\Delta_r} (\zeta u p(u))_\zeta \bar{u}\, dA_z = 0.$$

Using integration by parts we get

$$\int_{\Delta-\Delta_r} u_t\bar{u}\, dA - \frac{i}{2}\int_{\partial\Delta} \zeta|u|^2 p(u) d\bar{\zeta} + \frac{i}{2}\int_{\partial\Delta_r} \zeta|u|^2 p(u)\, d\bar{\zeta} = 0,$$

$$\int_{\Delta-\Delta_r} u_t\bar{u}\, dA - \frac{1}{2}\int_0^{2\pi} |u|^2 p(u)\, d\theta + \frac{1}{2}\int_0^{2\pi} r^2|u|^2 p(u)|_{\partial\Delta_r}\, d\theta = 0,$$

$$\frac{d}{dt}\int_{\Delta-\Delta_r} |u|^2\, dA - \int_0^{2\pi} d\theta + \int_{\Delta-\Delta_r} r^2|u|^2 p(u)|_{\partial\Delta_r}\, d\theta = 0.$$

On $\partial\Delta$, $\operatorname{Re} p(u) = 1/|u|^2$. We also know $\partial\bar{\partial}(\operatorname{Re} p(u)) = 0$, $\partial\bar{\partial}(1/|u|^2) \ge 0$, in Δ. Hence, in Δ, $\partial\bar{\partial}(1/|u|^2 - \operatorname{Re} p(u)) \ge 0$, which implies, by maximum principle, $1/|u|^2 - \operatorname{Re} p(u) \le 0$, i.e., $|u|^2 \operatorname{Re} p(u) \ge 1$.

Therefore, we get

$$\frac{d}{dt}\int_{\Delta-\Delta_r} |u|^2\, dA - \int_0^{2\pi} d\theta + \int_0^{2\pi} r^2\, d\theta \le 0,$$

$$\frac{d}{dt}\frac{1}{(1-r)}\int_{\Delta-\Delta_r} |u|^2\, dA \le 2\pi(1+r).$$

Letting $r \to 1$ we find

$$\frac{d}{dt}\int_{\partial\Delta} |u|^2\, d\theta \le 4\pi.$$

This proves the proposition.

It follows that if u is a solution of (2.1), then $u(.,t) \in H^2$, the Hardy space with power 2.

PROOF OF THEOREM 1.1.

(1) This is done in [2].

(2) Since $f(0,t) = 0$ we know, if

$$\begin{aligned} f(z,t) &= \sum_{n=1}^{\infty} a_n(t) z^n, \\ \|f(.,t)\|^2_{L^2_{\partial\Delta}} &= 2\pi \sum_{n=1}^{\infty} |a_n|^2 \\ &\le 2\pi \sum_{n=1}^{\infty} |na_n|^2 \\ &= \|u(.,t)\|^2_{L^2_{\partial\Delta}}. \end{aligned}$$

Hence, $\|f(.,t)\|^2_{W^{1,2}_{\partial\Delta}} \le Ct + C$, by the proposition.

Using Sobolev imbedding theorem on the unit circle, we find that

$$\|f(.,t)\| \leq (c_1 t + c_2)^{1/2},$$

where the norm is $C^{1/2}(\partial\Delta)$. This finishes the proof of Theorem 1.1.

We notice that a direct consequence of the theorem is the following well-known fact:

$$\|f(.,t)\|_\infty \leq (C_1 t + C_2)^{1/2}.$$

References

1. S. Richardson, Hele Shaw flows with a free boundary produced by the injection of fluid into a narrow channel, *Journal of Fluid Mechanics* **56**, 609–618 (1972).
2. U.P. Vinogradov, On a filter problem, *Prikladnaya Matematika i Makhanika* **12** (1948).
3. B. Gustafason, On a differential equation arising in a Hele Shaw moving boundary problem, *Arkive for Matematik* **22**, 251–268 (1984).
4. R.P. Gilbert and G.C. Wen, Free boundary problems occuring in planar fluid dynamics, *Nonlinear Analysis* **13**, 285–303 (1989).
5. O.A. Ladyzenskaja and N.N. Uraltseva, *Linear and Quasilinear Equation of Parabolic Type*, AMS Translation, (1968).